# DUALITY BETWEEN UNIFORM SPACES AND BOOLEAN ALGEBRAS

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Abstract. In this note we shall generalize the Stone duality between compact totally disconnected spaces and Boolean algebras to a duality between all complete non-Archimedean uniform spaces and Boolean algebras.

## 1. Boolean algebras

In this note, if $f : X \to Y$ is a function and $A \subseteq X, B \subseteq Y$, then we shall let $f''(A)$ be the image of $A$ and $f_{-1}(B)$ denote the inverse image of $B$.

Let $P$ be a poset. Then $x, y \in P$ are said to be incompatible if there does not exist an $r \in P$ with $r \leq x, r \leq y$. A subset $A \subseteq P$ is said to be cellular if every pair of elements in $A$ is incompatible.

If $(A, \wedge, 0)$ is a semilattice, then we shall say $x, y \in A \setminus \{0\}$ are incompatible if $x \wedge y = 0$, and we shall say $A' \subseteq A \setminus \{0\}$ is cellular if $x \wedge y = 0$ for $x, y \in A', x \neq y$.

**Theorem 1.1.** *Let $P$ be a poset. Then every cellular family is contained in a maximal cellular family (ordered under $\subseteq$).*

*Proof.* This is a simple application of Zorn's lemma. If $(R_b)_{b \in B}$ is a chain of cellular families, then $\cup_{b \in B} R_b$ is cellular. □

Let $P$ be a poset(semilattice), the write $c(P)$ for the collection of cellular families on $P$. If $A, B \in c(P)$, then write $A \preceq B$ if for each $a \in A$, there is a $b \in B$ with $a \leq b$. Since $B$ is cellular, the $b \in B$ with $a \leq b$ is unique, so let's write $\phi_{A,B} : A \to B$ for the unique function with $a \leq \phi_{A,B}(a)$.

**Theorem 1.2.** *$c(P)$ is a poset under the ordering $\preceq$, and $c(P)$ is an inverse system with the mappings $\phi_{A,B}$.*

*Proof.* Clearly $A \preceq A$, and for each $a \in A : \phi_{A,A}(a) = a$ since $a \leq a$. If $A \preceq B, B \preceq A$, then for each $a \in A$, we have $a \leq \phi_{A,B}(a) \leq \phi_{B,A}\phi_{A,B}(a)$, so since $A$ is cellular, we have $A$ be an antichain, so $a = \phi_{B,A}\phi_{A,B}(a)$, so $a = \phi_{A,B}(a)$, so $A \subseteq B$. Similarly, we have $B \subseteq A$, so $A = B$. Now if $A \preceq B, B \preceq C$, then $a \leq \phi_{A,B}(a) \leq \phi_{B,C}\phi_{A,B}(a)$, so $A \preceq C$, and $\phi_{A,C} = \phi_{B,C}\phi_{A,B}$. □

Let $B$ be a Boolean algebra. Then a partition $P$ of $B$ is a subset of $B \setminus \{0\}$ where $\vee P = 1$ and where $x \wedge y = 0$ for $x \neq y$. The partitions of a Boolean algebra are precisely the maximal elements of $c(B)$ with the inclusion ordering $\subseteq$. We shall write $\mathbb{P}(B)$ for the collection of all partitions of a Boolean algebra $B$.

**Lemma 1.3.** *For Boolean algebras $B$ the collection of partitions of $B$ form a lower semilattice where if $P, Q \in \mathbb{P}(B)$, then $P \wedge Q = \{p \wedge q | p \in P, q \in Q, p \wedge q \neq 0\}$.*

**Theorem 1.4.** *Let $p, q \in \mathbb{P}(B)$ and $p \preceq q$. Then if $p = \{a_i | i \in I\}, q = \{b_j | j \in J\}$, then for each $j \in J$ we have $b_j = \vee\{a_i | a_i \leq b_j\}$*

*Proof.* Let's assume that for some $j$ we do not have $b_j = \vee\{a_i | a_i \leq b_j\}$ , then there is an $r \in B$ with $r \geq a_i$ for $i \in I$ and $r < b_j$. We therefore have $b_j \wedge r' \neq 0$. If $a_i \leq b_j$, then $a_i \leq r$, so $(b_j \wedge r') \wedge a_i \leq (b_j \wedge r') \wedge r = 0$, and if $a_i \not\leq b_j$, then $a_i \leq b_{j'}$ for some $j' \neq j$, so $(b_j \wedge r') \wedge a_i \leq (b_j \wedge r') \wedge b_{j'} = 0$, so $(b_j \wedge r') \wedge a_i = 0$ for each $i \in I$, so $\{a_i | i \in I\}$ is not a partition of $B$. This is a contradiction. □

A partition $p$ on a Boolean algebra $B$ is said to be subcomplete if whenever $r \subseteq p$, $\vee r$ exists.

**Theorem 1.5.** *If a partition $p$ of a Boolean algebra $B$ is subcomplete and $p \preceq q$, then $q$ is subcomplete as well.*

*Proof.* Since $p \preceq q$ for each $a \in q$ there is a $P_a \subseteq p$ with $a = \vee P_a$. If $Q \subseteq q$, then $\vee(\cup_{a \in Q} P_a) = \vee_{a \in Q} \vee P_a = \vee Q$, so $q$ is subcomplete as well. □

**Theorem 1.6.** *Let $A$ be a subcomplete partition of a Boolean algebra $B$. Then the map $\phi : P(A) \to B$ given by $\phi(R) = \vee R$ is an injective Boolean algebra homomorphism with $\phi(\cup_{i \in I} R_i) = \vee_{i \in I} \phi(R_i)$ i.e. $\phi$ preserves all least upper bounds.*

*Proof.* Let $R_i \subseteq A$ for $i \in I$. Then $\phi(\cup_{i \in I} R_i) = \vee \cup_{i \in I} R_i = \vee_{i \in I} \vee R_i = \vee_{i \in I} \phi(R_i)$. Furthermore, $\phi(R) \vee \phi(R^c) = \phi(R \cup R^c) = \phi(A) = 1$ and $\phi(R) \wedge \phi(R^c) = (\vee R) \wedge (\vee R^c) = \vee_{a \in R, b \in R^c}(a \wedge b) = 0$, so $\phi(R^c) = \phi(R)^c$. Therefore $\phi$ is a Boolean algebra homomorphism preserving all least upper bounds, and $\phi$ is injective since $Ker(\phi)$ is trivial. □

A Boolean partition algebra is a pair $(B, F)$ where $B$ is a Boolean algebra and $F$ is a (possibly improper) filter on $\mathbb{P}(B)$ where $\{b, b'\} \in F$ for each $b \in B \setminus \{0, 1\}$.

**Lemma 1.7.** *Let $F \subseteq \mathbb{P}(B)$ be a filter. Then the following are equivalent.*
1. *$(B, F)$ is a Boolean partition algebra.*
2. *For each $b \in B \setminus \{0\}$ there is a $P \in F$ with $b \in P$.*
3. *$F$ contains all partitions of $B$ into finitely many sets.*

*Proof.* $1 \to 2$ Let $b \in B \setminus \{0\}$. If $b = 1$, then $b \in \{b\} \in F$. If $b \neq 1$, then $\{b, b'\} \in F$.

$2 \to 3$ Let $\{b_1, ..., b_n\}$ be a partition of $B$. If $n = 1$, then $\{b_1\} \in F$. If $n > 1$, then $\{b_1, ..., b_n\} \succeq \{b_1, b_1'\} \wedge ... \wedge \{b_n, b_n'\} \in F$.

$3 \to 1$ This is trivial. □

**Example 1.8.** If $B$ is a Boolean algebra, and $\lambda$ is an infinite cardinal, then define $\mathbb{P}_\lambda(B) = \{P \in \mathbb{P}(B) : |P| < \lambda\}$. Then $(B, \mathbb{P}_\lambda(B))$ is Boolean partition algebra.

**Theorem 1.9.** *If $B$ is a Boolean algebra, and $F \subseteq \mathbb{P}(B)$ is a filter, $\{0\} \cup \cup F$ is a subalgebra of $B$ and $(\{0\} \cup \cup F, F)$ is a Boolean partition algebra.*

*Proof.* Let $a, b \in \{0\} \cup \cup F$. If $a \wedge b = 0$, then $a \wedge b \in \{0\} \cup \cup F$. If $a \wedge b \neq 0$, then $a \neq 0, b \neq 0$, so there are $p, q \in F$ with $a \in p, b \in q$, so $a \wedge b \in p \wedge q$, so $a \wedge b \in \{0\} \cup \cup F$.

Clearly $0 \in \{0\} \cup \cup F$ and $1 \in \{1\}$, so $1 \in \{0\} \cup \cup F$.

If $a \in \{0\} \cup \cup F, a \neq 0, a \neq 1$, then $a \in p$ for some $p \in F$, so $a' \in \cup F$. Therefore $\{0\} \cup \cup F$ is a subalgebra of $B$.

Clearly $F$ is closed under $\wedge$. Now assume $p, q \in \mathbb{P}(\{0\} \cup \cup F), p \in F$ and assume $p \preceq q$. Then we claim that $q$ is a partition of $B$. Clearly $q$ is a cellular family. If

$x \in B$ and $x \geq b$ for each $b \in q$, then for each $a \in p$ we have a $b \in q$ with $a \leq b \leq x$, so $x = 1$. We therefore have $q$ be a partition of $B$, so $q \in F$. We therefore have $F$ be a filter on $\mathbb{P}(\{0\} \cup \cup F, F)$. If $a \in (\{0\} \cup \cup F) \setminus \{0\}$, then $a \in p \in F$ for some $p \in F$, so $(\{0\} \cup \cup F, F)$ is a Boolean partition algebra. $\square$

If $B$ is a Boolean algebra and $F$ is a filter on $\mathbb{P}(B)$, then write $\mathfrak{B}^*(B, F)$ for the Boolean partition algebra $(\{0\} \cup \cup F, F)$. If $B = P(X)$ for some set $X$, then we shall write $\mathfrak{B}^*(X, F)$ for $\mathfrak{B}^*(P(X), F)$.

Given a Boolean partition algebra $(B, F)$, and an ultrafilter $\mathcal{U} \subseteq B$, then we shall call $\mathcal{U}$ an $F$-ultrafilter if for each $P \in F$ there is an $a \in P \cap \mathcal{U}$ i.e. $|P \cap \mathcal{U}| = 1$ for each $P \in F$. We shall write $S_F(B)$ or $S^*(B, F)$ for the collection of all $F$-ultrafilters on $B$, and we shall write $S^*(B)$ for the collection of all ultrafilters on $B$.

**Lemma 1.10.** *Let $(B, F)$ be a Boolean partition algebra, and let $x = (x_p)_{p \in F} \in_{\leftarrow}^{Lim} F$. Then*

1. *If $p, q \in F$, then $x_{p \wedge q} = x_p \wedge x_q$*
2. *$b \in \{x_p | p \in F\}$ iff $b = 1$ or $x_{\{b, b'\}} = b$.*
3. *$\{x_p | p \in F\}$ is an $F$-ultrafilter on $B$.*

*Proof.* 1. We have $x_{p \wedge q} = a \wedge b$ for some $a \in p, b \in q$, so $a \wedge b = x_{p \wedge q} \leq x_p$ and $a \wedge b \leq x_q$. If $a \neq x_p$, then $a \wedge b = a \wedge b \wedge x_p = 0$ a contradiction. If $b \neq x_q$, then $a \wedge b = a \wedge b \wedge x_q = 0$ a contradiction. We therefore have $x_{p \wedge q} = a \wedge b = x_p \wedge x_q$.

2. $\leftarrow$ is trivial. For $\rightarrow$ assume $p \in F$. If $|p| = 1$, then $x_p = 1$. If $|p| > 1$, then let $q = \{x_p, x_p'\}$, the $p \preceq \{x_p, x_p'\} = q$, so $x_q = \phi_{p,q}(x_p) = x_p$.

3. Assume $p \in F$ and $x_p \leq a$ and $a \notin \{x_p | p \in F\}$. Then $x_{\{a, a'\}} = a'$, so $x_{p \wedge \{a, a'\}} = x_p \wedge x_{\{a, a'\}} = x_p \wedge a' = x_p \wedge a \wedge a' = 0$ a contradiction. We therefore have $\{x_p | p \in F\}$ be an upper set. If $p, q \in F$, then $x_p \wedge x_q = x_{p \wedge q}$, so $\{x_p | p \in F\}$ is a filter. If $b \in B \setminus \{0, 1\}$, then $x_{\{b, b'\}} = b$ or $x_{\{b, b'\}} = b'$, so $\{x_p | p \in F\}$ is an ultrafilter. $\{x_p | p \in F\}$ is an $F$-ultrafilter since $\{x_p | p \in F\} \cap p$ is nonempty for each $p \in F$. $\square$

Given $F$-ultrafilter $\mathcal{U}$, let $f : F \rightarrow B$ be the mapping where $f(p)$ is the unique element in $\mathcal{U} \cap p$. If $p \preceq q$, then $\phi_{p,q}(f(p)) \in q$ and $\phi_{p,q}(f(p)) \geq f(p) \in \mathcal{U}$, so $\phi_{p,q}(f(p)) = f(q)$. We therefore have $f \in_{\leftarrow}^{Lim} \mathcal{U}$.

Define maps $L :_{\leftarrow}^{Lim} F \rightarrow S_F(B), M : S_F(B) \rightarrow_{\leftarrow}^{Lim} F$ by letting $L(x_p)_{p \in F} = \{x_p | p \in F\}$ and where $M(\mathcal{U})(p) \in p \cap \mathcal{U}$ for $p \in F$.

**Theorem 1.11.** *The functions $L$ and $M$ are inverses.*

*Proof.* If $(x_p)_{p \in F} \in_{\leftarrow}^{Lim} F$, then for $p \in F$ we have $M(L((x_p)_{p \in F}))(p) = M(\{x_p | p \in F\})(p) = x_p$. Let $\mathcal{U}$ be an $F$-ultrafilter. If $a \in \mathcal{U}$, then let $p \in F$ be a partition with $a \in p$. Then $M(\mathcal{U})(p) = a$, so $a \in L(M(\mathcal{U}))$. We therefore have $\mathcal{U} \subseteq L(M(\mathcal{U}))$, so $\mathcal{U} = L(M(\mathcal{U}))$. $\square$

A Boolean partition algebra $(B, F)$ is said to be stable if for each $b \in B \setminus \{0\}$, there is an $(x_p)_{p \in F} \in_{\leftarrow}^{Lim} F$ and a $p \in F$ with $b = x_p$.

**Theorem 1.12.** *Let $(B, F)$ be a Boolean partition algebra, then the following are equivalent.*

1. *$(B, F)$ is stable.*
2. *The projections $\pi_p :_{\leftarrow}^{Lim} F \rightarrow p$ are all surjective.*
3. *$\cup S_F(B) = B \setminus \{0\}$*
4. *$\cap S_F(B) = \{1\}$*

*Proof.* $3 \leftrightarrow 4$. This is trivial.

$2 \rightarrow 1$ Let's assume that $\pi_p$ is surjective. Then for each $b \in B \setminus \{0\}$, there is a $p \in F$ with $b \in p$ and an $(x_p)_{p\in F} \in_{\leftarrow}^{Lim} F$ with $x_p = b$.

$3 \rightarrow 2$ Let's assume that $p \in F$. Then for each $b \in p$ there is a $\mathcal{U} \in S_F(B)$ with $b \in \mathcal{U}$, so $M(\mathcal{U})(p) = b$, so $\pi_p$ is surjective.

$1 \rightarrow 3$ Let's assume $(B, F)$ is stable, then for each $b \in B \setminus \{0\}$ there is an $(x_p)_{p\in F} \in_{\leftarrow}^{Lim} F$ and a $p \in F$ with $b = x_p$. We therefore have $b \in \{x_p | p \in F\} \in S^*(B, F)$. □

If $(P, \wedge, 0), (Q, \wedge, 0)$ are semilattices and $f : P \rightarrow Q$ is a semilattice homomorphism and $A \subseteq P \setminus \{0\}$ is a cellular family, then for each $a, b \in A, a \neq b$ we have $f(a) \wedge f(b) = f(a \wedge b) = f(0) = 0$, so $f''(A) \setminus \{0\}$ is a cellular family. If $(B, F)$ is a Boolean partition algebra, then write $\iota : (B, F) \rightarrow S^*(B, F)$ for the mapping where $\iota(a) = \{\mathcal{U} \in S^*(B, F) | a \in \mathcal{U}\}$. Then $\iota$ is a Boolean algebra homomorphism. If $p \in F$, then $\iota''(p) \setminus \{\emptyset\}$ is a partition of $S^*(B, F)$ for each $p \in F$. Moreover, $\iota$ is injective iff $Ker(\iota) = 0$ iff $(B, F)$ is stable. Moreover, if $(B, F)$ is stable, then since $\iota$ is injective, for $p, q \in F, p \neq q$ we have $\iota''(p) \neq \iota''(q)$.

**Theorem 1.13.** *If $(B, F)$ is a stable Boolean partition algebra, then for $p, q \in F$ we have $\iota''(p \wedge q) = \iota''(p) \wedge \iota''(q)$*

*Proof.* $\iota''(p \wedge q) = \iota''(\{a \wedge b | a \in p, b \in q\} \setminus \{0\}) = \iota''(\{a \wedge b | a \in p, b \in q\}) \setminus \{\emptyset\} = \{\iota(a \wedge b) | a \in p, b \in q\} \setminus \{\emptyset\} = \{\iota(a) \wedge \iota(b) | a \in p, b \in q\} \setminus \{\emptyset\} = \iota''(a) \wedge \iota''(b)$ □

Let $(A, F)$ be a Boolean partition algebra, and let $B$ be a Boolean algebra. Then a function $f : A \rightarrow B$ is partitional if $f$ is a Boolean algebra homomorphism, and $f''(p) \setminus \{0\}$ is a partition of $B$. A partition homomorphism $f : (A, F) \rightarrow (B, G)$ is a Boolean algebra homomorphism from $A$ to $B$ where $f''(p) \setminus \{0\} \in G$ for each $p \in F$. A function $f : (A, F) \rightarrow B$ if partitional iff $f : (A, F) \rightarrow (B, \mathbb{P}(B))$ is a partition homomorphism. If $f : (A, F) \rightarrow B$ is an injective homomorphism, then $f$ is partitional if and only if $f''(p)$ is a partition of $B$ for each $p \in F$. If $f : (A, F) \rightarrow (B, G)$ is an injective homomorphism, then $f$ is a partition homomorphism iff $f''(p) \in G$ for each $p \in F$.

**Theorem 1.14.** *1. Let $f : (A, F) \rightarrow (B, G)$ be a partition homomorphism, and let $g : (B, G) \rightarrow C$ be partitional. Then $g \circ f : (A, F) \rightarrow C$ is partitional as well.*

*2. Let $f : (A, F) \rightarrow (B, G), g : (B, G) \rightarrow (C, H)$ be partition homomorphism, then $g \circ f$ is also a partition homomorphism.*

*Proof.* In both case 1 and 2, we claim that $(g \circ f)''(p) \setminus \{0\} = g''(f''(p) \setminus \{0\}) \setminus \{0\}$. We have $(g \circ f)''(p) \setminus \{0\} = g''(f''(p)) \setminus \{0\} \supseteq g''(f''(p) \setminus \{0\}) \setminus \{0\}$. For the reverse inclusion, if $c \in (g \circ f)''(p) \setminus \{0\}$, then $c = g(b)$ for some $b \in f''(p)$, but since $c \neq 0$ we have $b \neq 0$ so $c = g(b) \in g''(f''(p) \setminus \{0\}) \setminus \{0\}$.

1. We have $f''(p) \setminus \{0\} \in G$, so $(g \circ f)''(p) \setminus \{0\} = g''(f''(p) \setminus \{0\}) \setminus \{0\}$ is a partition of $C$.

2. We have $f''(p) \setminus \{0\} \in G$, so $(g \circ f)''(p) \setminus \{0\} = g''(f''(p) \setminus \{0\}) \setminus \{0\} \in H$, so $g \circ f$ is a partition homomorphism. □

It can easily be seen that if $1 : (B, F) \rightarrow (B, F)$ is the identity mapping, then 1 a partition homomorphism. The class of all partition Boolean algebras therefore forms a category.

An extended partition is a family $(a_i)_{i\in I} \in B^I$ such that $\vee_{i\in I} a_i = 1$ and if $i \neq j$, then $a_i \wedge a_j = 0$. A family $(a_i)_{i\in I}$ is an extended partition iff $\{a_i | i \in I\}$ is a partition of $B$ and $a_i \neq a_j$ whenever $a_i \neq 0$.

**Theorem 1.15.**

*Let $A, B$ be Boolean algebras, then a function $f : A \to B$ is a Boolean algebra homomorphism iff whenever $(a, b, c)$ is an extended partition of $A$, then $(f(a), f(b), f(c))$ is an extended partition of $B$.*

*Proof.* $\Rightarrow$ Trivial.

$\Leftarrow$ First take note that since $(0, 0, 1)$ is an extended partition of $A$, we have $(f(0), f(0), f(1))$ be an extended partition of $B$, so $f(0) = f(0)\wedge f(0) = 0$. If $a \in A$, then $(a, a', 0)$ is an extended partition of $A$, so $(f(a), f(a'), f(0)) = (f(a), f(a'), 0)$ is an extended partition of $B$, so $f(a)' = f(a')$.

Assume $a, b \in A$ are incompatible, then $(a, b, (a \vee b)')$ is an extended partition of $A$, so $(f(a), f(b), f(a \vee b)')$ is an extended partition of $B$, so $f(a) \vee f(b) = f(a \vee b)$ and $f(a) \wedge f(b) = 0$.

Now assume $a \leq b$, then $f(b) = f((b \wedge a') \vee a) = f(b \wedge a') \vee f(a)$, so $f(a) \leq f(b)$.

Therefore for arbitrary $a, b \in B$ one has $f(a) \leq f(a \vee b), f(b) \leq f(a \vee b)$, so $f(a) \vee f(b) \leq f(a \vee b) = f((a \wedge b') \vee b) = f(a \wedge b') \vee f(b) \leq f(a) \vee f(b)$. Therefore $f$ is a Boolean algebra homomorphism. □

**Corollary 1.16.** *Let $(A, F)$ be a Boolean partition algebra, and let $B$ be a Boolean algebra. Then a function(not necessarily a Boolean algebra homomorphism) $f : (A, F) \to B$ is partitional iff $f(0) = 0$ and $(f(a))_{a\in p}$ is an extended partition of $B$.*

*Proof.* $f$ satisfies the hypothesis of theorem 1.15, so $f$ is a Boolean algebra homomorphism. □

## 2. Uniform Spaces and Duality

Given a set $X$, $\mathbb{P}(P(X))$ is the lattice of partitions on $X$. We shall write $\mathbb{P}P(X)$ for $\mathbb{P}(P(X))$. A uniform space $(X, F)$ is said to be non-Archimedean if it is generated by equivalence relations.

A partition space is a pair $(X, M)$ where $M$ is a filter on the lattice $\mathbb{P}P(X)$. We shall call the elements of $M$ crevasses. Partition spaces are essentially non-Archimedean uniform spaces, but in many circumstances partition spaces are easier to work with than non-Archimedean uniform spaces. We shall require every complete uniform space and complete partition space to be separating.

**Theorem 2.1.** *A separating partition space $(X, M)$ is complete iff whenever $\phi \in^{Lim}_{\leftarrow} M$, then there is an $x \in X$ with $x \in \phi(P)$ for each $P \in M$.*

*Proof.* $\to$ Let's assume $(X, M)$ is complete. Then let $\phi \in Lim_{\leftarrow} M$. Then $\{\phi(R)|R \in M\}$ is an ultrafilter on $\emptyset \cup \cup M$, so $\{\phi(R)|R \in M\}$ is a filterbase on $X$, and $\{\phi(R)|R \in M\}$ is clearly Cauchy. Since $(X, M)$ is complete, $\{\phi(R)|R \in M\}$ converges to some $x \in X$, so for each $P \in M$ we have $x \in \phi(P)$.

$\leftarrow$ Let $F$ be a Cauchy filter. Then for each $P \in M$, there is a unique $R \in P$ with $R \in F$. Let $\phi : M \to \mathfrak{B}^*(X, M)$ be the function with $\phi(P) \in P, \phi(P) \in F$ for each $P \in M$. If $P \preceq Q$, then $\phi_{P,Q}(\phi(P)) \in P$, and $\phi_{P,Q}(\phi(P)) \supseteq \phi(P) \in F$, so $\phi(Q) = \phi_{P,Q}(\phi(P))$. Therefore $\phi \in Lim_{\leftarrow} M$, so there is an $x \in X$ with $x \in \phi(P)$ for each $P \in M$. Therefore for each neighborhood $U$ of $x$, there is a $P \in M$ and an

$R \in P$ with $x \in R \subseteq U$, but we must have $R = \phi(P) \in F$, so $U \in F$ as well. We therefore conclude that $F \rightarrow x$. $\square$

**Theorem 2.2.** *Let $(B, F)$ be a stable Boolean partition algebra. Then*

*1. $\{\iota''(p)|p \in F\})$ generates a partition space structure on $S^*(B, F)$.*

*2. If $(B, F)$ is subcomplete, then $(S^*(B, F), \{\iota''(p)|p \in F\}))$ is a partition space when $S^*(B, F)$ is nonempty.*

*Proof.* 1. $\{\iota''(p)|p \in F\})$ is a filterbase since $\iota''(p) \wedge \iota''(q) = \iota''(p \wedge q)$.

2. Let's assume that $(B, F)$ is subcomplete. Then assume $p \in F$ and let $Z$ be a partition of $S^*(B, F)$ with $\iota''(p) \preceq Z$.

For each $R \in Z$, let $P_R = \{a \in p|\iota(a) \subseteq R\}$. Then let $r = \{\vee P_R|R \in Z\}$. Then $r$ is a partition of $B$ refining $p$, so $r$ must be subcomplete as well. If $\mathcal{U} \in R$, then $\mathcal{U} \in \iota(a)$ for some $a \in p$ with $\iota(a) \subseteq R$, so $a \in \mathcal{U}$, so $\vee P_R \in \mathcal{U}$, so $\mathcal{U} \in \iota(\vee P_R)$. We therefore have $R \subseteq \iota(\vee P_R)$, but since $\iota''(r) = \{\iota(\vee P_R)|R \in Z\}$ and $Z = \{R|R \in R\}$ are both partitions, we must have $\iota''(r) = Z \succeq p$. We therefore have $(S^*(B, F), \{\iota''(p)|p \in F\})$ be a partition space. $\square$

If $(B, F)$ is a Boolean partition algebra, then let $\psi : (B, F) \rightarrow \mathfrak{B}^*(S^*(B, F))$ be the mapping given by $\psi(x) = \iota(x)$. Given a partition space $(X, M)$, and $x \in X$, let $\mathcal{C}(x) = \{R \in \mathfrak{B}^*(X, M)|x \in R\}$, then $\mathcal{C}(x)$ is an ultrafilter on $\mathfrak{B}^*(X, M)$, and for each $P \in M$ there is a unique $R \in P$ with $x \in R$, so $R \in \mathcal{C}(x)$. We therefore have $\mathcal{C}(x) \in S^*(\mathfrak{B}^*(X, M))$ for each $x \in X$.

**Theorem 2.3.** *1. Let $(B, F)$ be a stable Boolean partition algebra, then $S^*(B, F)$ is a complete partition space.*

*2. If $(X, M)$ is a partition space, then $\mathfrak{B}^*(X, M)$ is a subcomplete and stable Boolean partition algebra.*

*3. Let $(B, F)$ be a stable Boolean partition algebra, then $\psi : (B, F) \rightarrow \mathfrak{B}^*(S^*(B, F))$ is an injective partition homomorphism, and if $(B, F)$ also subcomplete, then $\psi$ is a partition isomorphism.*

*4. Let $(X, M)$ be a partition space, then $\mathcal{C} : (X, M) \rightarrow S^*(\mathfrak{B}^*(X, M))$ is uniformly continuous, and $\mathcal{C}''(X, M)$ is dense in $S^*(\mathfrak{B}^*(X, M))$. If $(X, M)$ is separated, then $\mathcal{C}$ is a uniform embedding. If $(X, M)$ is complete, then $\mathcal{C}$ is a uniform homeomorphism.*

*Proof.* 1. To show that $S^*(B, F)$ is separated, assume $\mathcal{U}, \mathcal{V} \in S^*(B, F)$ are distinct ultrafilters. Then let $a \in \mathcal{U} \setminus \mathcal{V}$. Then $\{a, a'\} \in F$, but since $a \in \mathcal{U}, a' \in \mathcal{V}$ we have $\mathcal{U} \in \iota(a), \mathcal{V} \in \iota(a')$, so $\mathcal{U}, \mathcal{V}$ are in distinct blocks of the partition $\{\iota(a), \iota(a')\}$.

Let $M$ be the partition structure generated by $\{\iota''(p)|p \in F\}$, and let $\varphi \in_{\leftarrow}^{Lim} M$. Then for $p \in F$ we have $\iota''(p) \in M$, and $\varphi(\iota''(p)) \in \iota''(p)$, so let $x_p = \iota^{-1}(\varphi(\iota''(p)))$. Then $x_p \in p$ for $p \in F$. Moreover, if $p \preceq q$, then $\iota''(p) \preceq \iota''(q)$, so $\varphi(\iota''(p)) \subseteq \varphi(\iota''(q))$, so $x_p = \iota^{-1}(\varphi(\iota''(p))) \leq \iota^{-1}(\varphi(\iota''(q))) = x_q$, so $\varphi_{p,q}(x_p) = x_q$. We therefore have $(x_p)_{p \in F} \in_{\leftarrow}^{Lim} F$.

Let $\mathcal{U} = \{x_p|p \in F\}$. Then $\mathcal{U} \in S^*(B, F)$. Given $P \in M$ there is a $p \in F$ with $\iota''(p) \preceq P$, so since $x_p \in \mathcal{U}$ we have $\mathcal{U} \in \iota(x_p) = \varphi(\iota''(p)) \subseteq \varphi(P)$. We therefore have $S^*(B, F)$ be complete.

2. If $P \in M$ and $Z \subset P$ is non-empty, then $P \preceq \{\cup Z, \cup(P \setminus Z)\}$, so $\cup Z \in M$, so $\mathfrak{B}^*(X, M)$ is subcomplete.

To prove stability, assume $P \in M$. Then for each $R \in P$, let $x \in R$, then let $\varphi : M \rightarrow \cup M$ be the function where $x \in \varphi(Q) \in Q$ for each $Q \in M$. Then

$\varphi \in \overset{Lim}{\leftarrow} M$ and $\varphi(P) = R$. We therefore have the projection map $\pi_P : \overset{Lim}{\leftarrow} M \to P$ be surjective. We therefore have $\mathfrak{B}^*(X, M)$ be stable.

3. $\psi$ is injective since $Ker\psi = Ker\iota = \{0\}$. Let $M$ be the partition structure on $S^*(B, F)$. Then for $p \in F$ we have $\psi''(p) = \iota''(p) \in M$ for each $p \in F$, so $\psi$ is a partition homomorphism.

If $(B, F)$ is subcomplete, then $M = \{\iota''(p)|p \in F\} = \{\psi''(p)|p \in F\}$ and $B^*(S^*(B, F)) = B^*(S^*(B, F), M) = (\{\emptyset\} \cup \cup M, M)$, but $\{\emptyset\} \cup \cup M = \{\iota(b)|b \in B\} = \psi''(B)$, so $\psi$ is a partition isomorphism.

4. Take note that $\mathfrak{B}^*(X, M) = (\emptyset \cup \cup M, M)$, so $S^*(\mathfrak{B}^*(X, M)) = S^*(\emptyset \cup \cup M, M) = (S^*(\emptyset \cup \cup M, M), \{\iota''(P)|P \in M\})$ since $\mathfrak{B}^*(X, M)$ is subcomplete. To show $\mathcal{C}$ is uniformly continuous and dense we shall take inverse images of the partitions $\iota''(P)$ under $\mathcal{C}$. We have $\{\mathcal{C}_{-1}(R)|R \in \iota''(P)\} = \{\mathcal{C}_{-1}(\iota(V))|V \in P\}$. Now $x \in \mathcal{C}_{-1}(\iota(V))$ iff $\mathcal{C}(x) \in \iota(V)$ iff $V \in \mathcal{C}(x)$ iff $x \in V$, so $\mathcal{C}_{-1}(\iota(V)) = V$, so $\{\mathcal{C}_{-1}(R)|R \in \iota''(P)\} = \{\mathcal{C}_{-1}(\iota(V))|V \in P\} = \{V|V \in P\} = P$. We therefore have $\mathcal{C}$ be uniformly continuous, and since $\emptyset \notin \{\mathcal{C}_{-1}(R)|R \in \iota''(P)\}$ for each partition $\iota''(P)$, we have $\mathcal{C}''(X, M) \subseteq S^*(\mathfrak{B}^*(X, M))$ be dense.

If $(X, M)$ is separated, then one can clearly see that the function $\mathcal{C}$ is injective, so since each $P \in M$ can be written as $\{\mathcal{C}_{-1}(R)|R \in \iota''(P)\}$ we have $\mathcal{C}$ be an embedding. If $(X, M)$ is complete, then each $\mathcal{U} \in S^*(\mathfrak{B}^*(X, M)) = S^*(\emptyset \cup \cup M, M)$, so $\mathcal{U}$ is an ultrafilter with $|\mathcal{U} \cap P| = 1$ for each $P \in M$, so $\mathcal{U}$ is Cauchy, so $\mathcal{U} \to x$ for some $x \in X$. We therefore have $\mathcal{U} \subseteq \mathcal{C}(x)$, so $\mathcal{U} = \mathcal{C}(x)$. We therefore have $\mathcal{C}$ be surjective, so since $\mathcal{C}$ is a uniform embedding we have $\mathcal{C}$ be a uniform homeomorphism. □

If $(X, M)(Y, N)$ are uniform spaces, then a mapping $f : X \to Y$ is uniformly continuous iff $\{f_{-1}(R)|R \in Q\} \setminus \{\emptyset\} \in M$ for each $Q \in N$. If $f : X \to Y$ is uniformly continuous, then define a mapping $\mathfrak{B}^*(f) : \mathfrak{B}^*(Y, N) \to \mathfrak{B}^*(X, M)$ by letting $\mathfrak{B}^*(f)(R) = f_{-1}(R)$. Then clearly $\mathfrak{B}^*(f)$ is a partition homomorphism.

**Theorem 2.4.** *If $(A, F), (B, G)$ are Boolean partition spaces, and $\phi : A \to B$ is a partition homomorphism, then for each $\mathcal{U} \in S^*(B, F)$ we have $\phi_{-1}(\mathcal{U}) \in S^*(A, F)$*

*Proof.* Let's assume that $p \in F$. Then $\phi''(p) \setminus \{0\} \in G$, so there is an $a \in p$ where $\phi(a) \in \mathcal{U}$, so $a \in \phi_{-1}(\mathcal{U})$. □

For each pair of Boolean partition spaces $(A, F), (B, G)$ and partition homomorphism $\phi : (A, F) \to (B, G)$ define $S^*(\phi)$ by letting $S^*(\phi)(\mathcal{U}) = \phi_{-1}(\mathcal{U})$.

**Theorem 2.5.** *If $(A, F), (B, G)$ are stable Boolean partition algebras, and $\phi : (A, F) \to (B, G)$ is a partition homomorphism, then $S^*(\phi)$ is uniformly continuous.*

*Proof.* Let $P$ be a crevasse in $S^*(A, F)$. Then there is a $p \in F$ with $\iota''(p) \preceq P$. We shall take the inverse image of $\iota''(p)$ under $S^*(\phi)$. We have $\{S^*(\phi)_{-1}(R)|R \in \iota''(p)\} = \{S^*(\phi)_{-1}(\iota(r))|r \in p\}$. For $\mathcal{U} \in S^*(B, G)$ we have $\mathcal{U} \in S^*(\phi)_{-1}(\iota(r))$ iff $S^*(\phi)(\mathcal{U}) \in \iota(r)$ iff $r \in S^*(\phi)(\mathcal{U})$ iff $\phi(r) \in \mathcal{U}$ iff $\mathcal{U} \in \iota(\phi(r))$, so $S^*(\phi)_{-1}(\iota(r)) = \iota(\phi(r))$. We therefore have $\{S^*(\phi)_{-1}(R)|R \in \iota''(p)\} = \{S^*(\phi)_{-1}(\iota(r))|r \in p\} = \{\iota(\phi(r))|r \in p\} = \{\iota(s)|s \in \phi''(p)\}$. Therefore $\{S^*(\phi)_{-1}(R)|R \in \iota''(p)\} \setminus \{\emptyset\} = \{\iota(s)|s \in \phi''(p)\} \setminus \{\emptyset\} = \{\iota(s)|s \in \phi''(p) \setminus \{0\}\} = \iota''(\phi''(p) \setminus \{0\})$ be a crevasse in $S^*(B, G)$. We therefore have $S^*(\phi)$ be uniformly continuous. □

If $(X,L),(Y,M),(Z,N)$ are uniform spaces, and $f:X\to Y, g:Y\to Z$ are uniformly continuous continuous, then for $R\in\mathfrak{B}^*(Z,M)$ we have $\mathfrak{B}^*(g\circ f)(R)=(g\circ f)_{-1}(R)=f_{-1}(g_{-1}(R))=\mathfrak{B}^*(f)\circ\mathfrak{B}^*(g)(R)$. Furthermore, if $(A,F),(B,G),(C,H)$ are Boolean partition spaces, and $f:(A,F)\to(B,G), g:(B,G)\to(C,H)$ are partition homomorphisms, then $S^*(g\circ f)=(g\circ f)_{-1}(\mathcal{U})=f_{-1}\circ g_{-1}(\mathcal{U})=S^*(f)\circ S^*(g)(\mathcal{U})$. Therefore $\mathfrak{B}^*,S^*$ are contravariant functors since $\mathfrak{B}^*,S^*$ clearly map identity functions onto identity functions.

**Theorem 2.6.** *1. Let $(X,M),(Y,N)$ be uniform spaces, and let $f:X\to Y$ be uniformly continuous, then $S^*(\mathfrak{B}^*(f))\circ\mathcal{C}_{(X,M)}=C_{(Y,N)}\circ f$.*

*2. Let $(A,F),(B,G)$ be a stable Boolean algebras, and let $f:A\to B$ be a partition homomorphism, then $\mathfrak{B}^*(S^*(f))\psi_{(A,F)}=\psi_{(B,G)}f$.*

*3. If $(X,M)$ is a uniform space, then the functions $\mathfrak{B}^*(\mathcal{C}_X):\mathfrak{B}^*(S^*(\mathfrak{B}^*(X,M)))\to\mathfrak{B}^*(X,M)$ and $\psi:\mathfrak{B}^*(X,M)\to\mathfrak{B}^*(S^*(\mathfrak{B}^*(X,M)))$ are inverses.*

*4. If $(B,F)$ is a stable Boolean partition algebra, then $S^*(\psi_B):S^*(\mathfrak{B}^*(S^*(B,F)))\to S^*(B,F)$ and $\mathcal{C}:S^*(B,F)\to S^*(\mathfrak{B}^*(S^*(B,F)))$ are inverses.*

*Proof.* 1. For $x\in X$ we have $S^*(\mathfrak{B}^*(f))C_{(X,M)}(x)=\mathfrak{B}^*(f)_{-1}(\{R\in B^*(X,M)|x\in R\})=\{S\in B^*(Y,N)|x\in\mathfrak{B}^*(f)(S)\}=\{S\in\mathfrak{B}^*(Y,N)|x\in f_{-1}(S)\}=\{S\in\mathfrak{B}^*(Y,N)|f(x)\in S\}=\mathcal{C}_{(Y,M)}\circ f(x)$.

2. $\mathfrak{B}^*(S^*(f))\psi_A(a)=\mathfrak{B}^*(S^*(f))(\{\mathcal{U}\in S^*(A,F)|a\in\mathcal{U}\})=(S^*(f)_{-1}(\{\mathcal{U}\in S^*(A,F)|a\in\mathcal{U}\})=\{\mathcal{V}\in S^*(B,G)|a\in S^*(f)(\mathcal{V})\}=\{\mathcal{V}\in S^*(B,G)|a\in f_{-1}(\mathcal{V})\}=\{\mathcal{V}\in S^*(B,G)|f(a)\in\mathcal{V}\}=\psi_B\circ f(a)$.

3. We shall show that $\mathcal{B}^*(\mathcal{C})\psi:\mathfrak{B}^*(X,M)\to\mathcal{B}^*(X,M)$ is the identity function. If $R\in\mathfrak{B}^*(X,M)$, then $x\in\mathfrak{B}^*(\mathcal{C})\psi(R)=\mathcal{C}_{-1}\psi(R)$ iff $\mathcal{C}(x)\in\psi(R)$ iff $R\in\mathcal{C}(x)$ iff $x\in R$, so $\mathfrak{B}^*(\mathcal{C})\psi(R)=R$, so $\mathfrak{B}^*(\mathcal{C})$ is the identity function.

4. We shall show that $S^*(\psi)\mathcal{C}:S^*(B,F)\to S^*(B,F)$ is the identity function. Let $\mathcal{U}\in S^*(B,F)$. Then $a\in S^*(\psi)\mathcal{C}(\mathcal{U})$ iff $\psi(a)\in\mathcal{C}(\mathcal{U})$ iff $\mathcal{U}\in\psi(a)$ iff $a\in\mathcal{U}$. We therefore have $S^*(\psi)\mathcal{C}$ be the identity function. □

It should be noted that every compact space has a unique uniform structure. More specifically, if $X$ is compact, then let $F$ be the filter on $X\times X$ where $R\in F$ if $R$ is a neighborhood of the diagonal. Then $F$ is the unique uniformity on $X$. Furthermore, $(X,\mathcal{U})$ is a uniform space, then $X$ is compact iff $(X,\mathcal{U})$ is complete and totally bounded. The duality between compact totally disconnected spaces and Boolean algebras follows as a consequence of these facts.